

# Tangles of type $E_n$ and a reducibility criterion for the Cohen-Wales representation of the Artin group of type $E_6$

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**Abstract.** We introduce tangles of type  $E_n$  and construct a representation of the Birman-Murakami-Wenzl algebra (BMW algebra) of type  $E_6$ . As a representation of the Artin group of type  $E_6$ , this representation is equivalent to the faithful Cohen-Wales representation of type  $E_6$  that was used to show the linearity of the Artin group of type  $E_6$ . We find a reducibility criterion for this representation and complex values of the parameters for which the algebra is not semisimple.

## 1 Introduction and Main Results

Birman-Murakami-Wenzl algebras were introduced in [2] and independently in [22] in order to study the linearity of the braid groups. The BMW algebra has the same generators as those of the braid group and is a deformation of the Brauer centralizer algebra. Like the Brauer algebra, the BMW algebra has a diagrammatic version. It was shown by Morton and Wassermann in [21] to be isomorphic to the Morton-Traczyk algebra, an algebra of tangles. Moreover, the BMW algebra is equipped with a trace functional such that the Kauffman polynomial invariant of links is, after an appropriate renormalization that trace (see [2]). In [5], Cohen, Gijsbers and Wales generalize the BMW algebra to the simply laced Coxeter types  $D$  and  $E$  and in [6] they show that the BMW algebra of type  $D_n$  is isomorphic to a tangle algebra which they define. The generalized BMW algebra has the same generators as those of the Artin group in the same way the original BMW algebra has the same generators as those of the braid group. In [18], we use the tangle algebra defined in [5] to construct a representation of the BMW algebra of type  $D_n$   $BMW(D_n)$ , which as a representation of the Artin group  $\mathcal{A}(D_n)$  is equivalent to the generalized Lawrence–Krammer representation  $\mathcal{LK}(D_n)$  introduced by Arjeh Cohen and David Wales in [4]. We use this representation to deduce a reducibility criterion for  $\mathcal{LK}(D_n)$  as well as

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a conjecture which gives a criterion of semisimplicity for  $BMW(D_n)$ . In this paper we propose tangles of type  $E_6$ , and more generally  $E_n$ . We use them as a tool to construct a representation of the BMW algebra of type  $E_6$  (denoted  $BMW(E_6)$ ). As a representation of the Artin group of type  $E_6$ , this representation is equivalent to the generalized Lawrence–Krammer representation of type  $E_6$  (denoted  $\mathcal{LK}(E_6)$ ). We use our representation to give a reducibility criterion for  $\mathcal{LK}(E_6)$  and derive some complex values of the parameters for which  $BMW(E_6)$  is not semisimple.

In the following theorem, the representation  $\mathcal{LK}(E_6)$  renamed after its authors the Cohen–Wales representation of the Artin group of type  $E_6$  with parameters  $t$  and  $r$  is the one defined in [4].

**Theorem 1.** *The Cohen–Wales representation of the Artin group of type  $E_6$  with non-zero complex parameters  $t$  and  $r$  is irreducible, except when*

$$t \in \{1, -1, r^6, -r^{12}, r^{24}\},$$

*when it is reducible. Moreover,*

*when  $t = 1$ , there exists an invariant subspace of dimension 15.*

*When  $t = -1$ , there exists an invariant subspace of dimension 30.*

*When  $t = r^6$ , there exists an invariant subspace of dimension 20.*

*When  $t = -r^{12}$ , there exists an invariant subspace of dimension 6.*

*When  $t = r^{24}$ , there exists an invariant subspace of dimension 1.*

**Theorem 2.** *Let  $l, m$  and  $r$  be three non-zero complex numbers with  $m = \frac{1}{r} - r$ . When*

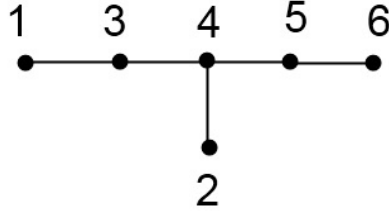
$$l \in \left\{ r^3, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}, -\frac{1}{r^9}, r^9, \frac{1}{r^{21}}, -r^{21} \right\}$$

*the BMW algebra of type  $E_6$  with non-zero complex parameters  $l$  and  $m$  is not semisimple over the field  $\mathbb{Q}(l, r)$ .*

The interest of the paper is double. The Lawrence–Krammer representations have become of interest since the original Lawrence–Krammer representation was used by Stephen Bigelow [1] and independently Daan Krammer [12] to show the long open problem of the linearity of the braid group. The generalized Lawrence–Krammer representations of types  $D$  and  $E$  introduced by Cohen and Wales were in turn used by them to show the linearity of the Artin groups of these types (see [4]). Other constructions and proof of linearity can also be found in [8]. Reducibility criteria for these representations exist in types  $A$  and  $D$  (see [15] and [18] respectively). In each case, reducibility is shown for some complex specializations of the two parameters of the representation, while the representations are generically irreducible (see [5], [20], [26], [15] in type  $A$  and [5] and [18] in type  $D$ ). More work allows to determine the complete structure of the representation. This was done in type  $A$  in [24] and independently in [17] by different means; it was later achieved in type  $D$  in

[19]. The specializations for which the representation becomes reducible give non-semisimplicity for the BMW algebra whose parameters are related to the parameters of the representation. This is another important aspect. There has been a long interest in studying the semisimplicity of the BMW algebras. The problem in type  $A$  originated in Result (a) of [25], was further developed in Theorem 2 of [16] and was completely solved in Theorem B of [23], building upon [9]. The problem in type  $D$  is stated as a conjecture in [18] and is to this date not solved.

We end this introduction by recalling the defining relations of  $BMW(E_6)$ . First, we shall draw a Dynkin diagram of type  $E_6$ .



We write  $i \sim j$  if nodes  $i$  and  $j$  are adjacent on the Dynkin diagram and  $i \not\sim j$  if nodes  $i$  and  $j$  are not adjacent on the Dynkin diagram. The BMW algebra of type  $E_6$  with nonzero complex parameters  $l$  and  $m$  is an algebra over  $\mathbb{Q}(l, m)$  with six generators called the  $g_i$ 's,  $1 \leq i \leq 6$ . It contains other elements, namely the  $e_i$ 's,  $1 \leq i \leq 6$  that are related to the  $g_i$ 's as below.

$$\begin{aligned}
 (A1) \quad g_i g_j &= g_j g_i && \text{if } i \not\sim j \\
 (A2) \quad g_i g_j g_i &= g_j g_i g_j && \text{if } i \sim j \\
 (P) \quad e_i &= \frac{l}{m} (g_i^2 + m g_i - 1) && \text{for all } i \\
 (DL1) \quad g_i e_i &= \frac{1}{l} e_i && \text{for all } i \\
 (DL2) \quad e_i g_j e_i &= l e_i && \text{when } i \sim j
 \end{aligned}$$

$(A_1)$  and  $(A_2)$  are the Artin group relations;  $(P)$  defines each  $e_i$  as a polynomial in  $g_i$ ;  $(DL1)$  and  $(DL2)$  are called the delooping relations because they are the algebraic versions of the delooping relations on the tangles (see § 2, point (iii) of Definition 2.6 of [6]).

Some consequences of these defining relations are the following.

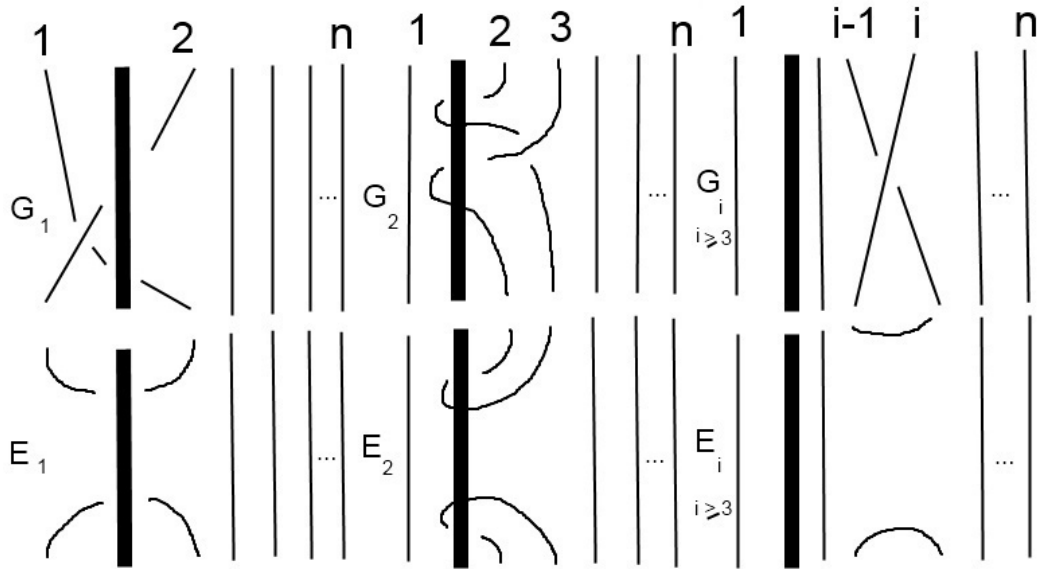
$$\begin{aligned}
 (I) \quad e_i^2 &= \delta e_i && \text{for all } i \text{ where } \delta = 1 - \frac{l-1}{m} \\
 (MA) \quad g_i g_j e_i &= e_j e_i = e_j g_i g_j && \text{when } i \sim j \\
 (R) \quad e_i e_j e_i &= e_i && \text{when } i \sim j
 \end{aligned}$$

$(I)$  expresses the fact that  $\delta^{-1} e_i$  is an idempotent; We call equalities in  $(MA)$  "mixed Artin relations";  $(R)$  can be viewed as a reduction.

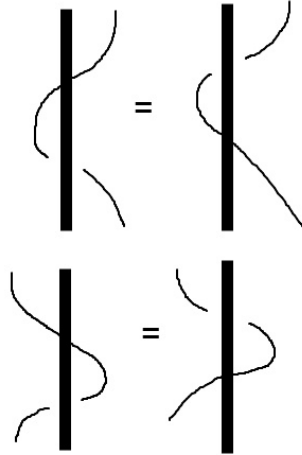
A result of [7] states that  $BMW(E_6)$  is semisimple and free over the integral domain  $\mathbb{Z}[\delta, \delta^{-1}, l, l^{-1}, m]/(m(1-\delta) - (l - \frac{1}{l}))$  of rank 1, 440, 585 and that it is cellular in the sense of Graham and Lehrer [10] over suitable rings.

## 2 Tangles of type $E_n$

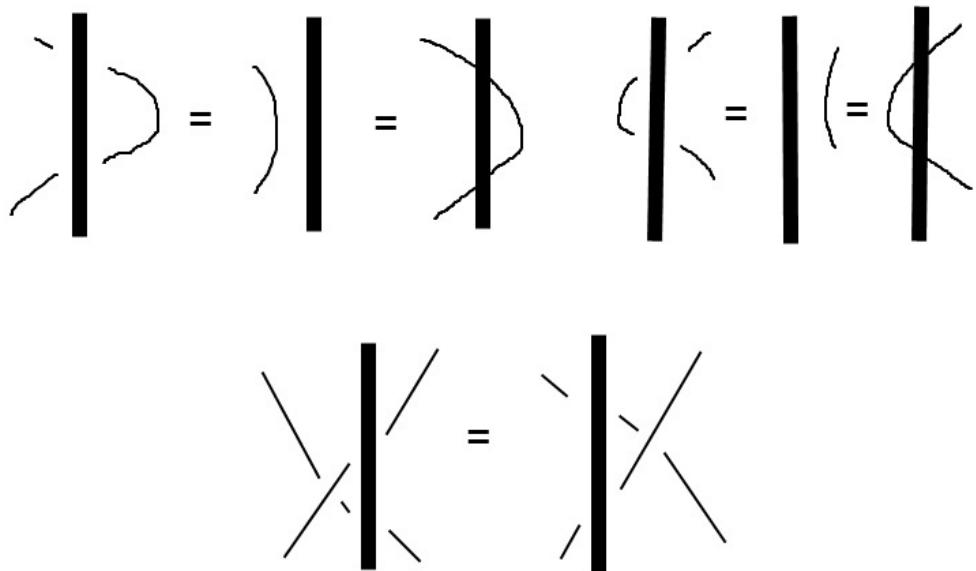
We use tangles as a tool to construct a representation of  $BMW(E_6)$ . Here is how the algebra elements  $e_i$ 's and  $g_i$ 's are represented in terms of tangles.



The rigid bar is called the pole. The strands can twist around the pole. A strand twists around the pole if it goes behind the pole and then over it, or the converse. An important aspect is that for a strand twisting around the pole, the following relations are satisfied.

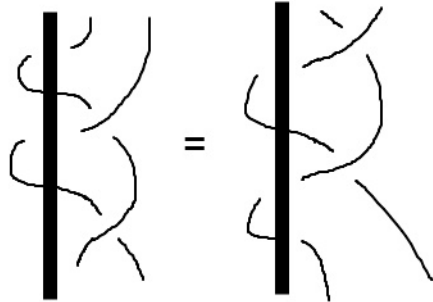


They are called the twist relations. We impose that the tangles satisfy other relations, some of which are given in [6]. We add to the classical Reidemeister's moves II and III two pole-related Reidemeister's moves that we call Reidemeister's moves of types IV and V. They will be respectively denoted by (R IV) and (R V).



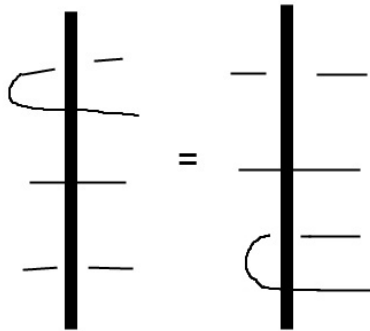
*The pole-related Reidemeister's moves*

We now recall below the commuting relation of [6].



First commuting relation

We add a new commuting relation which is specific to type  $E_6$  and which is the following



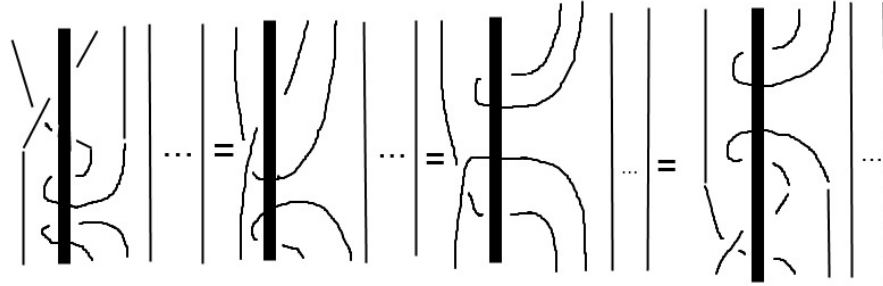
Second commuting relation

We call it "commuting relation of the second type" and we refer to it as (C II). We call the commuting relation of [6] "commuting relation of the first kind" and we will refer to it as (C I). It is a straightforward verification to check that by using the twist relation, (R IV) and (C II), the following relations are satisfied

on the tangles

$$\begin{aligned} G_1 E_2 &= E_2 G_1 \\ E_1 G_2 &= G_2 E_1 \\ G_1 G_2 &= G_2 G_1 \\ E_1 E_2 &= E_2 E_1 \end{aligned}$$

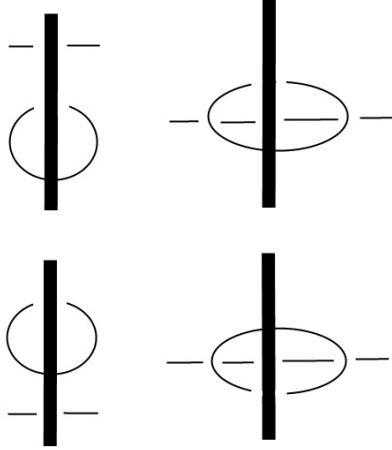
We do the first one and leave the other ones to the reader.



The first and third equalities are obtained by using (R IV); the second equality is obtained by using (CII) in the following way. Divide the middle tangle to the left into three vertical strips. The strip in the middle contains the twist, the horizontal over-crossing and the horizontal under-crossing. The strands in the right hand side strip intersect the border of the middle strip in four points. The strands in the left hand side strip intersect the border of the middle strip in two points. Apply (CII) in the middle strip with the extremities of the strands reaching the points mentioned above in the order in which they appear, while leaving the other strips unchanged. Obtain the middle tangle to the right.

Some closed pole loop relations are given in Proposition 2.10 of [6]. Here, we show an additional proposition.

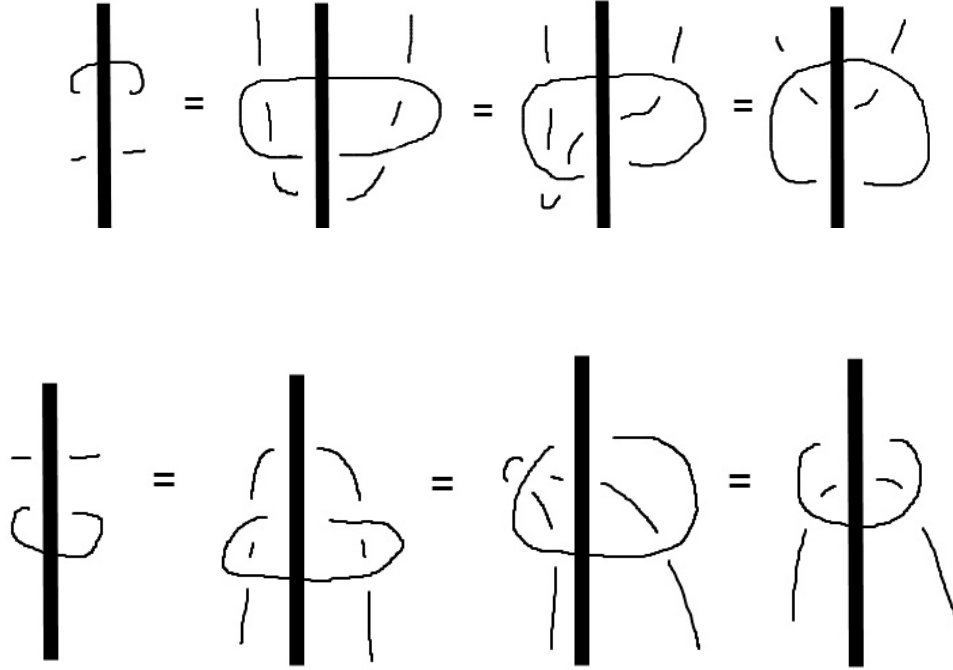
**Proposition 1.** *The tangles of type  $E_6$  satisfy the following closed pole loop relations that we call "the fourth closed pole loop relations".*



*The fourth closed pole loop relations.*

PROOF. We partly use the same argument as in the proof of Proposition 2.10 of [6], point (i). Indeed, using (R II), we enlarge the closed loop in such a way that it over-crosses the horizontal strand four times. We then apply (R V), followed by (R II). Contrary to [6], it is not the commuting relation that plays a role here, but the new Reidemeister's move (R V). This property is thus specific to tangles of type  $E_6$ . The moves are illustrated on the figures below.

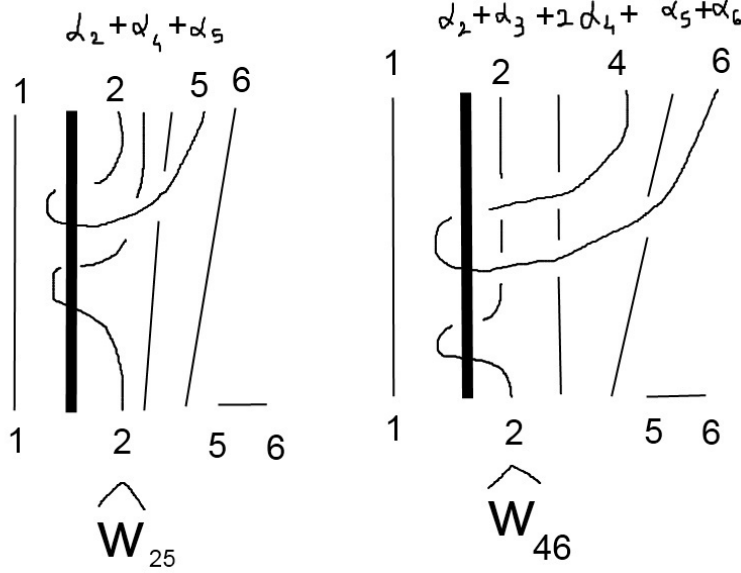




For all the other relations satisfied on the tangles, namely the Kauffman skein relation, the self-intersection relations, the idempotent relation, the first pole-related self-intersection relation, the second pole-related self-intersection relation, the first closed pole loop relation and their consequences, we refer the reader to Definition 2.6 and to Figures 6, 9, 13 of [6]. With these relations, all the defining relations of the BMW algebra are satisfied on the tangles  $E_i$ 's and  $G_i$ 's. Thus, there is a morphism of algebras between the BMW algebra and the tangle algebra, which sends  $g_i$  to  $G_i$  and  $e_i$  to  $E_i$ . We do not know whether this morphism is an isomorphism of algebras and do not answer the question in this paper. This homomorphic image of the BMW algebra in the tangle algebra is likely the whole tangle algebra, but we do not consider this in our paper. We simply use the tangles as a tool in order to get the actions in the representation that we build. We then check with Maple that the map that we define is indeed a representation of the BMW algebra.

### 3 The representation

We build a representation of  $BMW(E_6)$  inside the vector space  $V_6$  over  $\mathbb{Q}(l, r)$  spanned by vectors indexed by the 36 positive roots of a root system of type  $E_6$ . By definition,  $r$  and  $-\frac{1}{r}$  are the two non-zero complex roots of the polynomial  $X^2 + mX - 1$ , and so  $m$  and  $r$  are related by  $m = \frac{1}{r} - r$ . For the vectors indexed by the positive roots, we introduce some convenient notations. First, if we forget about node 1 and the edge joining nodes 1 and 3 on the Dynkin diagram, we obtain a Dynkin diagram of type  $D_5$  on nodes 2, 3, 4, 5, 6. For the vectors indexed by the positive roots issued from this diagram, we use the same notations as in [18] with node  $i + 1$  now playing the role of node  $i$ . So, for instance  $\widehat{w_{2j}}$ ,  $j \geq 4$  (resp  $\widehat{w_{23}}$ ) denotes the vector associated with the positive root  $\alpha_2 + \alpha_4 + \dots + \alpha_j$  (resp with the simple root  $\alpha_2$ ); we denote by  $\widehat{w_{3j}}$ ,  $j \geq 4$  the vector associated with the positive root  $\alpha_2 + \alpha_3 + \alpha_4 + \dots + \alpha_j$ ; we denote by  $\widehat{w_{st}}$ ,  $s \geq 4$  the vector associated with the positive roots  $\alpha_2 + \alpha_3 + 2\alpha_4 + \dots + 2\alpha_s + \alpha_{s+1} + \dots + \alpha_t$ . Like we did in type  $D$ , we associate tangles to these vectors as on the examples below. And so, to each of these vectors corresponds an algebra element on which the  $g_i$ 's can act to the left.



In what follows, by  $\widehat{w_{st}}$  we mean, depending on the context, the algebra element or the tangle representing the algebra element or the basis vector itself. By abuse of terminology, we will also sometimes speak indifferently of the positive root, the vector indexed by it, the algebra element associated with it or the tangle representing the algebra element. Using these conventions, we now present notations for the vectors indexed by a positive root whose support contains node number 1.

**Definition 1.** *We define*

$$\begin{aligned}
\widehat{w_{st}^{\Delta^+}} &= g_1 \widehat{w_{st}} & \text{when } s \geq 3 \\
\widehat{w_{st}^{\Delta^{++}}} &= g_3 \widehat{w_{st}^{\Delta^+}} & \text{when } s \geq 4 \\
\widehat{w_{56}^{\Delta^{+++}}} &= g_4 \widehat{w_{56}^{\Delta^{++}}} \\
\widehat{w_{56}^{\Delta^l}} &= g_2 \widehat{w_{56}^{\Delta^{+++}}}
\end{aligned}$$

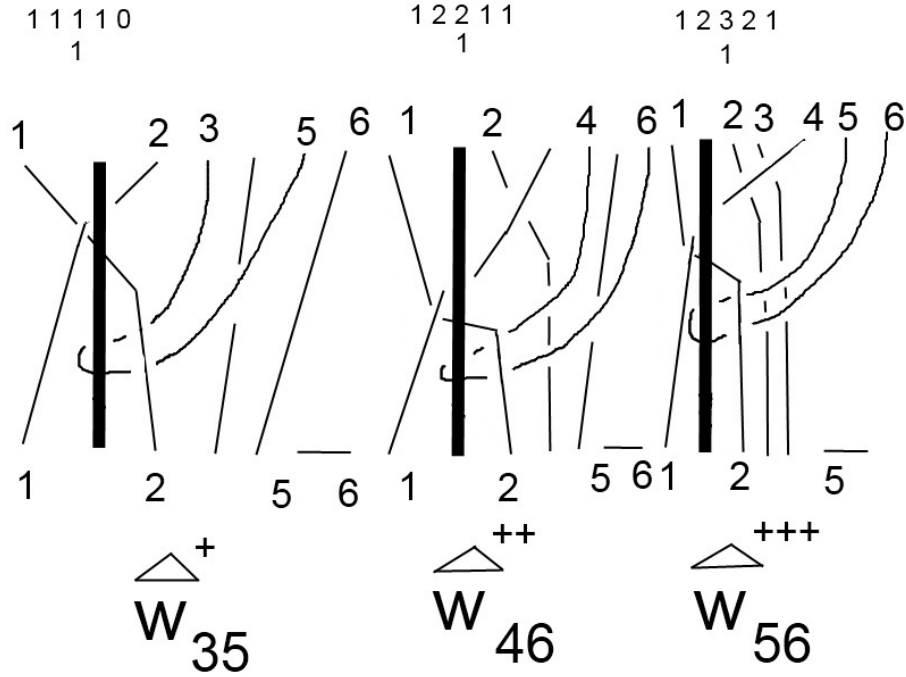
Following the notations of [3], we denote by

$$\begin{array}{cccccc}
a & c & d & e & f & \\
& & & b & & 
\end{array}$$

the positive root  $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6$ . And so, we see that

$$\text{A vector indexed by } \beta \text{ carries } \left\{ \begin{array}{l} \text{a hat if and only if } a = 0 \text{ and } b = 1 \\ \text{a triangle if and only if } a = 1 \text{ and } b \neq 0. \\ \text{a triangle and an additive sign if and only if } a = b = c = 1. \\ \text{a triangle and two additive signs if and only if } c = d = 2 \\ \text{a triangle and three additive signs if and only if } b = 1 \text{ and } d = 3 \\ \text{a triangle and an "l" as in "longest root" if and only if } b = 2 \end{array} \right.$$

Examples of tangles representing some elements of Definition 1 are given below, where we used the Bourbaki notation to refer to the positive root on the top line.



Note the number of additive signs gives the number of crossings on the vertical strands. To obtain these diagrams from those representing the  $\widehat{w}_{ij}$ 's with  $i \geq 3$ , we use the first commuting relation. This relation allows to switch the order of the two strands that twist around the pole by putting one on top of the other and conversely. The vertical strand that twists around the pole used to under-cross the horizontal strand twisting around the pole. After using the first commuting relation, it over-crosses it. Changing the order of these two strands then allows to use the pole-related Reidemeister's move of type (IV). For clarity, we now list in the table below the 36 positive roots of type  $E_6$  and the vectors of  $V_6$  indexed by them.

$\alpha_1$	$w_{12}$
$\alpha_1 + \alpha_3 + \cdots + \alpha_t$	$w_{1t}, 3 \leq t \leq 6$
$\alpha_{i+1} + \cdots + \alpha_j$	$w_{ij}, 2 \leq i < j \leq 6$
$\alpha_2$	$\widehat{w_{23}}$
$\alpha_2 + \alpha_4 + \cdots + \alpha_t$	$\widehat{w_{2t}}, 4 \leq t \leq 6$
$\alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_t$	$\widehat{w_{3t}}, t \geq 4$
$\alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$	$\widehat{w_{st}}, 4 \leq s < t \leq 6$
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_t$	$\Delta^+ w_{3t}, t \geq 4$
$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$	$\Delta^+ w_{st}, 4 \leq s < t \leq 6$
$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_s + \alpha_{s+1} + \cdots + \alpha_t$	$\Delta^{++} w_{st}, 4 \leq s < t \leq 6$
$\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\Delta^{+++} w_{56}$
$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	$\Delta^l w_{56}$

The algebra elements associated with the positive roots are gathered in the following table and in Definition 1.

$e_i \dots e_6$	$w_{i-1,i}, i \geq 3$
$e_1 e_3 e_4 e_5 e_6$	$w_{12}$
$g_j \dots g_{i+2} e_{i+1} \dots e_6$	$w_{ij}, j \geq i+2 \text{ and } i \neq 1$
$g_t \dots g_3 e_1 e_3 e_4 e_5 e_6$	$w_{1t}, t \geq 3$
$e_2 e_4 e_5 e_6$	$\widehat{w_{23}}$
$g_j \dots g_4 e_2 e_4 e_5 e_6$	$\widehat{w_{2j}}, j \geq 4$
$g_i \dots g_3 g_j \dots g_4 e_2 e_4 e_5 e_6$	$\widehat{w_{ij}}, 3 \leq i < j \leq 6$

Let  $F$  denote the ground field  $\mathbb{Q}(l, r)$  and  $B(E_6)$  the BMW algebra of type  $E_6$  formerly denoted by  $BMW(E_6)$ . Our representation is built inside the  $B(E_6)$ -module

$$B(E_6)e_6 \Big/ \left( \langle B(E_6)e_i e_j B(E_6) \rangle_{i \neq j} \cap B(E_6)e_6 \right) \otimes_{\mathcal{H}} F,$$

where  $\mathcal{H}$  is the Hecke algebra of type  $A_5$  with generators  $g_2, g_4, g_3, g_1$  and  $z$ . These generators satisfy the relations  $(A_1), (A_2)$  from the defining Artin relations of the BMW algebra, the relations  $g_i^2 + m g_i = 1$  for all  $i$ , the relation  $z^2 + m z = 1$  and the relations

$$\begin{cases} z g_i &= g_i z & \text{for all } i \in \{2, 3, 4\} \\ g_1 z g_1 &= z g_1 z \end{cases}$$

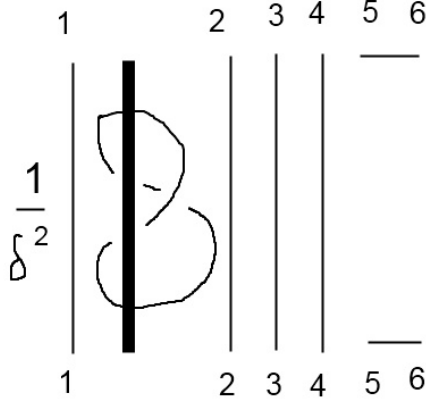
We denote by  $M$  the left  $B(E_6)$ -module

$$M = B(E_6)e_6 \Big/ \left( \langle B(E_6)e_i e_j B(E_6) \rangle_{i \neq j} \cap B(E_6)e_6 \right)$$

We provide  $M$  with a structure of right  $\mathcal{H}$ -module by the following actions. The  $g_i$ 's act to the right of elements of  $M$  by simply multiplying them to the right by  $g_i$  in  $M$ . The Hecke algebra element  $z$  acts to the right of an element of  $M$  by multiplying it to the right in  $M$  by

$$\xi = \frac{1}{\delta^2} e_6 e_5 e_4 e_2 g_3 e_2 e_4 e_5 e_6$$

The tangle representing  $\xi$  is the following.



Note that applying the commuting relation of the second type on  $\xi$  leaves the tangle invariant. It is a result of [6] that the  $(0, 0)$ -tangle that has the shape of an eight, called  $\Xi^+$  in [6] commutes with any twist around the pole, as illustrated on Figure 9 of [6]. Then, by making the tangle  $\Xi^+$  as small as we want, we see that  $\Xi^+$  commutes with  $g_2$ . Hence  $\xi$  commutes with  $g_2$ . A quick glance at the geometric definition of  $\xi$  shows that  $\xi$  also commutes with  $g_3$  and  $g_4$ . We now show that  $\xi$  and  $g_1$  satisfy the following relation.

**Proposition 2.**  $g_1 \xi g_1 = \xi g_1 \xi$

PROOF. Follows from the relations (I) and (R) and from the Artin relation  $g_1 g_3 g_1 = g_3 g_1 g_3$ .

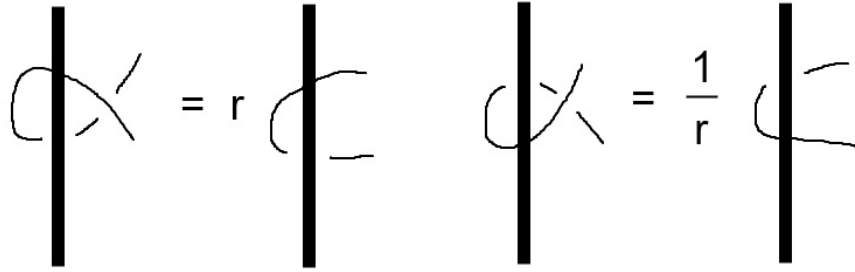
The rest of the proof that  $M$  is a right  $\mathcal{H}$ -module for the action provided above is up to an appropriate change in the indices similar to the proof of Claim 1 of [18]. Recall that  $r^2 + m r = 1$ . We then provide the ground field  $F$  with a structure of left  $\mathcal{H}$ -module by letting the generators of  $\mathcal{H}$  act on 1 in the following

way.

$$\begin{cases} g_i.1 &= r & \text{for all } i \\ z.1 &= r \end{cases}$$

We obtain a left representation of  $BMW(E_6)$  by considering the tensor product  $M \otimes_{\mathcal{H}} F$ .

We now use the tangles as a tool to compute the left actions of the  $g_i$ 's on the basis vectors of  $V_6$  inside the left  $BMW(E_6)$ -module  $M \otimes_{\mathcal{H}} F$ . The fact that we work inside  $M \otimes_{\mathcal{H}} F$  allows to multiply a tangle at the bottom by  $g_i$  (resp  $g_i^{-1}$ ) for any  $i \in \{1, 2, 3, 4\}$  at the cost of a multiplication (resp division) by  $r$ . It also allows to replace a pole related self intersection by a simple twist around the pole at the cost of a factor  $r$  or  $\frac{1}{r}$ , depending on the sign of the crossing (see Figure 9 of [6] and the explanations in § 2.2 of [18]). The move is illustrated on the figure below.



Suppressing a self-intersection around the pole

Fig 1

Finally it allows to suppress a pole-related self intersection as on Figure 13 of [6] at the cost of a multiplication by  $r$ . We succeed to define a representation of  $BMW(E_6)$  inside  $V_6$ . We define a map

$$\nu^{(6)} : \begin{array}{ccc} BMW(E_6) & \longrightarrow & \text{End}_F(V_6) \\ g_i & \longmapsto & \nu_i \end{array} ,$$

with the endomorphisms  $\nu_i$ 's given below.

First, we give the left action by  $g_1$  on our basis vectors. When it not described, the action on the omitted basis vectors is a multiplication by  $r$ .

$$\nu_1(w_{12}) = \frac{1}{l} w_{12} \quad (1)$$

$$\forall j \geq 3, \quad \nu_1(w_{1j}) = w_{2j} \quad (2)$$

$$\forall j \geq 3, \quad \nu_1(w_{2j}) = w_{1j} + m r^{j-3} w_{12} - m w_{2j} \quad (3)$$

$$\forall j \geq 3, \quad \nu_1(\widehat{w_{2j}}) = r \widehat{w_{2j}} \quad (4)$$

$$\forall s \geq 3, \quad \nu_1(\widehat{w_{sj}}) = \widehat{w_{sj}}^+ \quad (5)$$

$$\forall t \geq 4, \quad \nu_1(\widehat{w_{3t}}^+) = \widehat{w_{3t}} - m \widehat{w_{3t}}^+ + \frac{m}{l} r^{t-6} w_{12} \quad (6)$$

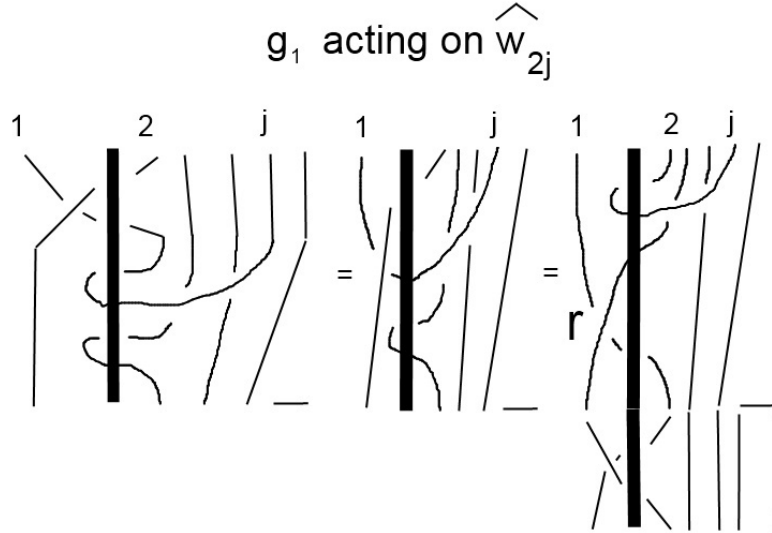
$$\forall s \geq 4, \quad \nu_1(\widehat{w_{sj}}^+) = \widehat{w_{sj}} - m \widehat{w_{sj}}^+ + \frac{m}{l} r^{s+t-9} w_{12} \quad (7)$$

$$\forall s \geq 4, \quad \nu_1(\widehat{w_{sj}}^{\Delta^{++}}) = r \widehat{w_{sj}}^{\Delta^{++}} \quad (8)$$

$$\nu_1(\widehat{w_{56}}^{\Delta^{+++}}) = r \widehat{w_{56}}^{\Delta^{+++}} \quad (9)$$

$$\nu_1(\widehat{w_{56}}^{\Delta^l}) = r \widehat{w_{56}}^{\Delta^l} \quad (10)$$

Equality (4) is obtained by using the second commuting relation as on the figure below.



To derive equality (6), we must compute the action of  $e_1$  on  $\widehat{w_{3t}}$ . To do so, we use the tangles. We apply the commuting relation of the first kind and a Reidemeister's move of type IV. Then, we apply the commuting relation of the second kind. We multiply the result at the bottom by  $g_1$  at the cost of a division by  $r$ . Then we apply (R V) to move the crossing issued from  $g_1$  to the right hand



side of the pole. Next, we successively apply (R IV) and (R II). We then remove the crossings between the vertical strands at the cost of a factor  $r^{t-4}$ . Finally, we apply (C II) a second time. When doing so, we create a pole-related self-intersection using the terminology of [6]. As explained in [18] we can replace it with a simple twist around the pole by dividing by a factor  $r$ . Hence the term in  $\frac{m}{l}r^{t-6}w_{12}$  in (6). Some of the moves have been gathered on the figure below.

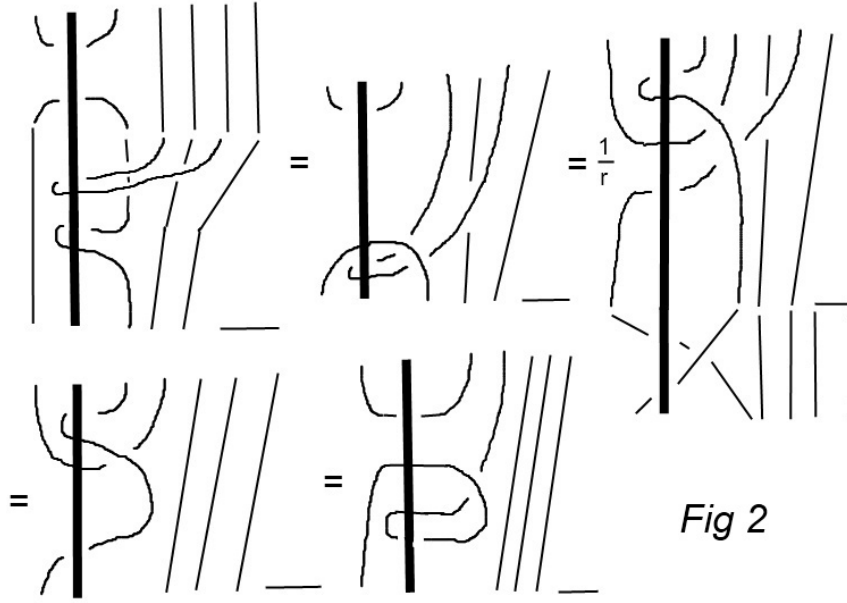


Fig 2

Equality (7) is now obtained from Equality (6) as follows.

$$\begin{aligned}
 \forall s \geq 4, \quad e_1 \widehat{w_{st}} &= e_1 g_{s,4} \widehat{w_{3t}} \\
 &= g_{s,4} e_1 \widehat{w_{3t}} \\
 &= r^{t-6} g_{s,4} w_{12} \\
 &= r^{s+t-9} w_{12}
 \end{aligned}$$

Equalities (8), (9) and (10) follow from the Artin group relations and from the fact that when  $s \geq 4$ , we have  $g_3 \widehat{w_{st}} = r \widehat{w_{st}}$ .

The action by  $g_2$  on the  $\widehat{w_{st}}$ 's with  $2 \leq s \leq n-1$  and  $3 \leq t \leq n$  is given by reading the action  $\nu_1(\widehat{w_{s-1,t-1}})$  on the formulas (1), (2), (3), (6) and (7) of Theorem 1 of [18] and by incrementing all the indices of the terms in  $w$ 's by one while leaving the exponents unchanged. The action by  $g_2$  on the terms in  $w_{2j}$ 's,  $j \geq 3$  (resp  $w_{3j}$ 's,  $j \geq 4$ ) are given by reading the actions  $\nu_1(w_{1,j-1})$ 's (resp  $\nu_1(w_{2,j-1})$ 's) on the formula (5) (resp (4)) of Theorem 1 of [18] and by incrementing all the indices of the terms in  $w$ 's by one while leaving the expo-

nents unchanged. Next, we define

$$\nu_2(w_{12}) = r w_{12} \quad (11)$$

$$\nu_2(w_{13}) = r w_{13} \quad (12)$$

$$\forall j \geq 4, \quad \nu_2(w_{1j}) = m \widehat{w_{3j}} + \overset{\Delta^+}{w_{3j}} - m r^{j-6} w_{12} + m r^{j-5} \left( \frac{1}{r} \widehat{w_{23}} - w_{13} \right) + m \left( \frac{1}{r} \widehat{w_{2j}} - w_{1j} \right) \quad (13)$$

$$\nu_2(\overset{\Delta^+}{w_{3t}}) = w_{1t} - m w_{2t} - m r w_{3t} + m r^{t-5} (w_{12} + r w_{13} - m r w_{23}) \quad (14)$$

$$\forall s \geq 4, \quad \nu_2(\overset{\Delta^+}{w_{st}}) = \begin{cases} m r^{t-6} (r \widehat{w_{2s}} + m w_{2s} - w_{1s}) \\ + m r^{s-4} (r \widehat{w_{2t}} + m w_{2t} - w_{1t}) \\ + m r^{t-5} (\overset{\Delta^+}{w_{3s}} - r w_{3s}) \\ + m r^{s-3} (\overset{\Delta^+}{w_{3t}} - r w_{3t}) \\ + m^2 (r^{s+t-10} + r^{s+t-8}) (r \widehat{w_{23}} + m w_{23} - w_{13} - \frac{1}{r} w_{12}) \\ + r \overset{\Delta^+}{w_{st}} \end{cases} \quad (15)$$

$$\forall s \geq 4, \quad \nu_2(\overset{\Delta^{++}}{w_{st}}) = \begin{cases} m r^{t-5} \left( \widehat{w_{3s}} + r \overset{\Delta^+}{w_{3s}} + m \left( r + \frac{1}{r} \right) w_{3s} - (w_{1s} + r w_{2s}) \right) \\ + m r^{s-3} \left( \widehat{w_{3t}} + r \overset{\Delta^+}{w_{3t}} + m \left( r + \frac{1}{r} \right) w_{3t} - (w_{1t} + r w_{2t}) \right) \\ + m^2 (r^{s+t-10} + r^{s+t-8}) \left( r^2 (\widehat{w_{23}} - w_{23}) - (w_{12} + r w_{13}) \right) \\ + r \overset{\Delta^{++}}{w_{st}} \end{cases} \quad (16)$$

$$\nu_2(\overset{\Delta^{+++}}{w_{56}}) = \overset{\Delta^l}{w_{56}} \quad (17)$$

$$\begin{aligned} \nu_2(\overset{\Delta^l}{w_{56}}) &= \overset{\Delta^{+++}}{w_{56}} - m \overset{\Delta^l}{w_{56}} \\ &+ \left[ \frac{m}{lr} + \frac{(1-r^4)^2}{l} + \frac{m(1-r^4)(1-r^6)}{r^2} \right] \widehat{w_{23}} \end{aligned} \quad (18)$$

When computing the action by  $g_2$  on  $w_{12}$  with the tangles, one gets the same tangle as on the bottom line of Figure 2 (the tangle on the left hand side), except the top horizontal strand over-crosses the vertical strand that it intersects. After applying the commuting relation of the second type, the resulting pole-related

self-intersection has the opposite sign than the one on Figure 2. By Figure 1, it yields a multiplication by  $r$  instead of a division by  $r$ . We thus obtain (11). From (11), we easily derive (12) by using  $w_{13} = g_3 w_{12}$  and by using the fact that  $g_2$  and  $g_3$  commute.

To get equality (13), we use  $w_{1j} = g_1^{-1} w_{2j}$ . It follows that

$$\begin{aligned}
g_2 w_{1j} &= g_2 g_1^{-1} w_{2j} \\
&= g_1^{-1} g_2 w_{2j} \\
&= g_1^{-1} (\widehat{w_{3j}} + m r^{j-5} (\widehat{w_{23}} - w_{23}) + m (\widehat{w_{2j}} - w_{2j})) \\
&= m \widehat{w_{3j}} + \widehat{w_{3j}}^{\Delta^+} - m r^{j-6} w_{12} + m r^{j-5} \left( \frac{1}{r} \widehat{w_{23}} - w_{13} \right) + m \left( \frac{1}{r} \widehat{w_{2j}} - w_{1j} \right)
\end{aligned}$$

As for equality (14), we have

$$\begin{aligned}
g_2 \widehat{w_{3t}}^{\Delta^+} &= g_2 g_1 \widehat{w_{3t}} \\
&= g_1 g_2 \widehat{w_{3t}} \\
&= g_1 (w_{2t} + m r^{t-4} w_{23} - m w_{3t}) \\
&= w_{1t} + m r^{t-3} w_{12} - m w_{2t} + m r^{t-4} (w_{13} + m w_{12} - m w_{23}) - m r w_{3t} \\
&= w_{1t} - m w_{2t} - m r w_{3t} + m r^{t-5} (w_{12} + r w_{13} - m r w_{23})
\end{aligned}$$

To derive Equality (15), we use again  $g_2 g_1 \widehat{w_{st}} = g_1 g_2 \widehat{w_{st}}$ . Then, we apply formula (1) of [18] on  $\nu_1(\widehat{w_{s-1,t-1}})$  and increment all the indices of the result by one while leaving the exponents unchanged. We thus get an expression for  $g_2 \widehat{w_{st}}$ . To finish, we apply formulas (3), (4), (5) of the current paper.

Applying  $g_3$  on the result of (15) now yields (16). The action by  $g_3$  on  $\widehat{w_{3t}}^{\Delta^+}$  appears later when  $\nu_3$  is defined and is simply a multiplication by  $r$  (see (20)).

To compute (18), we proceed as follows. First, we use the Kauffman skein relation and we get

$$g_2 \widehat{w_{56}}^{\Delta^l} = \widehat{w_{56}}^{\Delta^{+++}} - m \widehat{w_{56}}^{\Delta^l} + \frac{m}{l} e_2 g_4 g_3 g_1 \widehat{w_{56}}$$

It remains to compute  $e_2 g_4 g_3 g_1 \widehat{w_{56}}$ . We have

$$\begin{aligned}
e_2 g_4 g_3 g_1 \widehat{w_{56}} &= e_2 g_4 g_2 g_1^{-1} \widehat{w_{56}}^{\Delta^{++}} \\
&= e_2 e_4 g_2^{-1} \widehat{w_{56}}^{\Delta^{++}}
\end{aligned}$$

To compute  $g_2^{-1} \widehat{w_{56}}^{\Delta^{++}}$ , we use the formulas (11), (12), (13), (16) above and the formulas (2) – (7) of [18]. The computations are long. We content ourselves to give the result but show on an example how they work. In the following equalities, the symbol  $\blacklozenge$  denotes the action for the representation defined in Theorem 1 of [18] and the symbol  $\langle\langle\rangle\rangle$  is the operation that increments the indices by one while leaving the exponents unchanged. We will also use these

notations whenever we need them later along the paper. So for instance, we have

$$\begin{aligned}
g_2^{-1} w_{15} &= g_2^{-1} g_1^{-1} w_{25} \\
&= g_1^{-1} g_2^{-1} w_{25} \\
&= g_1^{-1} \langle\langle g_1^{-1} \blacklozenge w_{14} \rangle\rangle \\
&= g_1^{-1} \langle\langle \widehat{w_{24}} - m w_{12} + m \widehat{w_{14}} \rangle\rangle \\
&= g_1^{-1} (\widehat{w_{35}} - m w_{23} + m \widehat{w_{25}}) \\
&= \widehat{w_{35}} + m \widehat{w_{35}} - \frac{m}{r} w_{12} - m w_{13} + \frac{m}{r} \widehat{w_{25}}
\end{aligned}$$

By doing such computations, we get

$$\begin{aligned}
g_2^{-1} \widehat{w_{56}}^{\Delta^{++}} = & \frac{1}{r} \left[ \widehat{w_{56}}^{\Delta^{++}} - m^2 (r + r^3) (r^2 l \widehat{w_{23}} - r w_{23} - \frac{1}{r} w_{12} - w_{13}) \right. \\
& - m r^2 \left( \begin{aligned} & r w_{16} + r^2 w_{26} - m (1 + r^2) w_{36} - r \widehat{w_{36}} \\ & - m l r^3 (1 + r^2) \widehat{w_{23}} + m r^2 (1 + r^2) w_{23} \\ & + m (1 + r^2) w_{12} + m r (1 + r^2) w_{13} - r^2 \widehat{w_{36}}^{\Delta^+} \end{aligned} \right) \\
& \left. - m r \left( \begin{aligned} & r w_{15} + r^2 w_{25} - m (1 + r^2) w_{35} - r \widehat{w_{35}} \\ & - m l r^2 (1 + r^2) \widehat{w_{23}} + (1 - r^4) w_{23} \\ & + m (\frac{1}{r} + r) w_{12} + m (1 + r^2) w_{13} - r^2 \widehat{w_{35}}^{\Delta^+} \end{aligned} \right) \right] \quad (19)
\end{aligned}$$

Next, we must act on this result with  $e_2 e_4$ . The actions are summarized below.

**Lemma 1.** *The following equalities hold.*

$$\begin{array}{llll}
e_2 e_4 w_{16} & = & 0 & \\
e_2 e_4 w_{26} & = & 0 & \\
e_2 e_4 w_{36} & = & l r \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{36}} & = & \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{36}}^{\Delta^+} & = & r \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{36}}^{\Delta^{++}} & = & \widehat{w_{23}} & \\
e_2 e_4 w_{15} & = & 0 & \\
e_2 e_4 w_{25} & = & 0 & \\
e_2 e_4 w_{35} & = & l \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{35}} & = & \frac{1}{r} \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{35}}^{\Delta^+} & = & \widehat{w_{23}} & \\
e_2 e_4 w_{12} & = & 0 & \\
e_2 e_4 w_{13} & = & \frac{1}{r} \widehat{w_{23}} & \\
e_2 e_4 w_{23} & = & \widehat{w_{23}} & \\
e_2 e_4 \widehat{w_{23}} & = & \widehat{w_{23}} &
\end{array}$$

PROOF. In most cases, we use the tangles with

$$e_2 e_4 = \text{Diagram}$$

In some cases it is easier to proceed algebraically like below.

$$\begin{aligned} e_2 e_4 \widehat{w_{36}}^{\Delta^+} &= e_2 e_4 g_1 \widehat{w_{36}} \\ &= g_1 e_2 e_4 \widehat{w_{36}} \\ &= g_1 \widehat{w_{23}} \\ &= r \widehat{w_{23}} \end{aligned}$$

By (A1) and (P), the computation reduces to computing  $e_2 e_4 \widehat{w_{36}}$ , which is easily computed with the tangles and is simply  $\widehat{w_{23}}$ . The last equality then holds by (4). Of course this equality can also be visualized by using the tangles. Also,

$$\begin{aligned} e_2 e_4 w_{12} &= e_2 (e_4 e_1) e_3 e_4 e_5 e_6 \\ &= 0 \text{ in } M \end{aligned}$$

A more tricky computation is the following.

$$\begin{aligned} e_2 e_4 \widehat{w_{56}}^{\Delta^{++}} &= e_2 e_4 g_3 g_1 \widehat{w_{56}} \\ &= e_2 e_4 g_3 g_4 g_4^{-1} g_1 \widehat{w_{56}} \\ &= e_2 e_4 e_3 g_1 g_4^{-1} \widehat{w_{56}} \\ &= \frac{1}{r} e_2 e_4 e_3 \widehat{w_{56}}^{\Delta^+} \end{aligned}$$

Then, by using

$$e_3 = 1 + \frac{1}{m}(g_3 - g_3^{-1}),$$

and forthcoming equalities (21) and (22), we get

$$e_3 \widehat{w_{56}}^{\Delta^+} = r \widehat{w_{23}}$$

And since  $e_2 e_4 w_{23} = \widehat{w_{23}}$ , we get  $e_2 e_4 \widehat{w_{56}}^{\Delta^{++}} = \widehat{w_{23}}$ , as in the lemma. We are now ready to apply  $e_2 e_4$  on the right hand side of (19). We get

$$e_2 e_4 g_2^{-1} w_{56}^{\Delta^{++}} = \left[ \frac{1}{r} + \left( \frac{1}{r^2} - r^2 \right) \left( l(1 - r^6) + r^3 + r^5 \right) \right] \widehat{w_{23}}$$

It remains to multiply by  $\frac{m}{l}$  and it yields the last term of (18).

We now move on to  $\nu_3$ . The action by  $g_3$  on the terms  $w_{ij}$ 's with  $i \geq 2$  is the same action as in the representation of [18]. The action by  $g_3$  on the terms  $\widehat{w_{ij}}$ 's is obtained as follows. Read the action of  $g_2$  on the term  $\widehat{w_{i-1,j-1}}$  using formulas (8) – (10), (12), (13), (16) of [18], then increment all the indices of the result by one without modifying the exponents. The other actions are given by

$$\nu_3(w_{3t}^{\Delta^+}) = r w_{3t}^{\Delta^+} \quad (20)$$

$$\forall s \geq 4, \quad \nu_3(w_{st}^{\Delta^+}) = w_{st}^{\Delta^{++}} \quad (21)$$

$$\forall s \geq 4, \quad \nu_3(w_{st}^{\Delta^{++}}) = w_{st}^{\Delta^+} - m w_{st}^{\Delta^{++}} + \frac{m}{l} r^{s+t-10} w_{23} \quad (22)$$

$$\nu_3(w_{56}^{\Delta^{+++}}) = r w_{56}^{\Delta^{+++}} \quad (23)$$

$$\nu_3(w_{56}^{\Delta^l}) = r w_{56}^{\Delta^l} \quad (24)$$

To get equality (20), we notice that

$$\begin{aligned} g_3 w_{3t}^{\Delta^+} &= g_3 g_1 g_3 \widehat{w_{2t}} \\ &= g_1 g_3 g_1 \widehat{w_{2t}} \\ &= r g_1 \widehat{w_{3t}} \\ &= r w_{3t}^{\Delta^+} \end{aligned}$$

To get Equality (22), apply the Kauffman skein relation and get the first two terms in (22), plus a multiple of  $e_3 g_1 \widehat{w_{st}}$ . We compute the latter term as follows. We have

$$\begin{aligned} e_3 g_1 \widehat{w_{st}} &= e_3 g_1 g_3 g_3^{-1} \widehat{w_{st}} \\ &= \frac{1}{r} e_3 e_1 \widehat{w_{st}} \\ &= \frac{1}{r} e_3 (r^{s+t-9} w_{12}) \\ &= r^{s+t-10} w_{23} \end{aligned}$$

Equality (24) is obtained by using the sequence of relations

$$\begin{aligned}
g_3 \widehat{w_{56}^{\Delta^l}} &= g_3 g_2 g_4 g_3 g_1 \widehat{w_{56}} \\
&= g_2 g_3 g_4 g_3 g_1 \widehat{w_{56}} \\
&= g_2 g_4 g_3 g_4 g_1 \widehat{w_{56}} \\
&= g_2 g_4 g_3 g_1 g_4 \widehat{w_{56}} \\
&= r \widehat{w_{56}^{\Delta^l}}
\end{aligned}$$

We go on with  $\nu_4$ . We only provide below the actions that don't easily follow from the  $D_5$  case in [18] or which are the most difficult.

$$\forall t \in \{5, 6\}, \quad \nu_4(\widehat{w_{4t}^{\Delta^+}}) = \widehat{w_{3t}^{\Delta^+}} + \frac{m r^{t-5}}{l} w_{34} - m \widehat{w_{4t}^{\Delta^+}} \quad (25)$$

$$\forall t \in \{5, 6\}, \quad \nu_4(\widehat{w_{4t}^{\Delta^{++}}}) = r \widehat{w_{4t}^{\Delta^{++}}} \quad (26)$$

$$\nu_4(\widehat{w_{56}^{\Delta^{+++}}}) = \widehat{w_{56}^{\Delta^{+++}}} - m \widehat{w_{56}^{\Delta^{+++}}} + \frac{m}{l} w_{34} \quad (27)$$

$$\nu_4(\widehat{w_{56}^{\Delta^l}}) = \left\{ \begin{array}{l} m^2(r + r^3) \left( \begin{array}{l} w_{45} - r w_{35} + r(w_{46} - r w_{36}) \\ -r^2[w_{14} - r w_{13} + r(w_{24} - r w_{23})] \\ -w_{34} \end{array} \right) \\ + m r^2 (\widehat{w_{45}} - r \widehat{w_{35}} + r (\widehat{w_{45}^{\Delta^+}} - r \widehat{w_{35}^{\Delta^+}} + \widehat{w_{46}} - r \widehat{w_{36}}) + r^2 (\widehat{w_{46}^{\Delta^+}} - r \widehat{w_{36}^{\Delta^+}})) \\ + m^2 r^2 (1 + r^2)^2 (\widehat{w_{24}} - r \widehat{w_{23}}) \\ + r \widehat{w_{56}^{\Delta^l}} \end{array} \right. \quad (28)$$

We now comment on these equalities. Equality (25) is done by using the fact that  $g_1$  and  $g_4$  commute.

Equality (26) is derived by using a trick already used before.

$$\begin{aligned}
g_4 \widehat{w_{4t}^{\Delta^{++}}} &= g_4 g_3 g_4 g_4^{-1} g_1 \widehat{w_{4t}} \\
&= g_3 g_4 g_3 g_1 g_4^{-1} \widehat{w_{4t}} \\
&= g_3 g_4 g_3 \widehat{w_{3t}^{\Delta^+}} \\
&= r g_3 g_4 \widehat{w_{3t}^{\Delta^+}} \\
&= r g_3 \widehat{w_{4t}^{\Delta^+}} \\
&= r \widehat{w_{4t}^{\Delta^{++}}}
\end{aligned}$$

For the last term of (27), we use the mixed Artin relation (MA) like we did

when we computed  $\nu_3(\overset{\Delta^{++}}{w_{st}})$ . This computation is left to the reader. Finally, we have

$$\begin{aligned}
g_4 \cdot \overset{\Delta^l}{w_{56}} &= g_4 g_2 g_4 \cdot \overset{\Delta^{++}}{w_{56}} \\
&= g_2 g_4 g_2 \cdot \overset{\Delta^{++}}{w_{56}} \\
&= g_2 \cdot \left\{ m r \left( \widehat{w_{45}} + r \overset{\Delta^+}{w_{45}} + m \left( \frac{1}{r} + r \right) w_{45} - r w_{15} - r^2 w_{25} \right) \right. \\
&\quad \left. + m r^2 \left( \widehat{w_{46}} + r \overset{\Delta^+}{w_{46}} + m \left( r + \frac{1}{r} \right) w_{46} - r w_{16} - r^2 w_{26} \right) \right. \\
&= \quad \left. + m^2 (r + r^3) \left( r^2 (\widehat{w_{24}} - w_{24}) - r w_{12} - r w_{14} \right) \right. \\
&\quad \left. + r \overset{\Delta^{+++}}{w_{56}} \right\}
\end{aligned}$$

To compute the term in the last equality, we used the matrix of the left action by  $g_2$  and Maple. We obtained (28).

Again, for  $\nu_5$ , we only provide here the actions that are the less straightforward. The following equalities hold.

$$\nu_5(\overset{\Delta^+}{w_{35}}) = \overset{\Delta^+}{w_{34}} + \frac{m r}{l} w_{45} - m \overset{\Delta^+}{w_{35}} \quad (29)$$

$$\nu_5(\overset{\Delta^+}{w_{56}}) = \overset{\Delta^+}{w_{46}} + \frac{m}{l} w_{45} - m \overset{\Delta^+}{w_{56}} \quad (30)$$

$$\nu_5(\overset{\Delta^{++}}{w_{45}}) = r \overset{\Delta^{++}}{w_{45}} \quad (31)$$

$$\nu_5(\overset{\Delta^{++}}{w_{46}}) = \overset{\Delta^{++}}{w_{56}} \quad (32)$$

$$\nu_5(\overset{\Delta^{++}}{w_{56}}) = \overset{\Delta^{++}}{w_{46}} + \frac{m r}{l} w_{45} - m \overset{\Delta^{++}}{w_{56}} \quad (33)$$

$$\nu_5(\overset{\Delta^{+++}}{w_{56}}) = r \overset{\Delta^{+++}}{w_{56}} \quad (34)$$

$$\nu_5(\overset{\Delta^l}{w_{56}}) = r \overset{\Delta^l}{w_{56}} \quad (35)$$



The following sets of equalities explain (33), (34) and (35) respectively.

$$\begin{aligned}
\nu_5(\overset{\Delta^{++}}{w_{56}}) &= g_5 g_3 \overset{\Delta^+}{w_{56}} \\
&= g_3 g_5 \overset{\Delta^+}{w_{56}} \\
&= g_3(\overset{\Delta^+}{w_{46}} + \frac{m}{l} w_{45} - m \overset{\Delta^+}{w_{56}}) \\
&= \overset{\Delta^{++}}{w_{46}} + \frac{mr}{l} w_{45} - m \overset{\Delta^{++}}{w_{56}}
\end{aligned}$$

$$\begin{aligned}
g_5(\overset{\Delta^{+++}}{w_{56}}) &= g_5 g_4 \overset{\Delta^{++}}{w_{56}} \\
&= g_5 g_4 g_5 g_5^{-1} \overset{\Delta^{++}}{w_{56}} \\
&= g_4 g_5 g_4 \overset{\Delta^{++}}{w_{46}} \\
&= g_4 g_5 (r \overset{\Delta^{++}}{w_{46}}) \\
&= r g_4 \overset{\Delta^{++}}{w_{56}} \\
&= r \overset{\Delta^{+++}}{w_{56}}
\end{aligned}$$

$$\begin{aligned}
g_5(\overset{\Delta^l}{w_{56}}) &= g_5 g_2 \overset{\Delta^{+++}}{w_{56}} \\
&= g_2 g_5 \overset{\Delta^{+++}}{w_{56}} \\
&= r g_2 \overset{\Delta^{+++}}{w_{56}} \\
&= r \overset{\Delta^l}{w_{56}}
\end{aligned}$$

The action by  $g_6$  is in many ways similar to the one by  $g_5$  and is left to the reader. We use in particular the formulas (13) and (15) of [18] with  $j = 6$ .

With Maple, we formed the matrices of the  $\nu_i$ 's and checked that they verify all the Artin relations. Further, we define  $\nu(e_i) = \frac{l}{m}(\nu_i^2 + m \nu_i - id_{V_6})$  and we check that the matrices of the  $\nu(e_i)$ 's and the matrices of the  $\nu_i$ 's satisfy the delooping relations (DL1) and (DL2). We thus obtain a representation of the BMW algebra of type  $E_6$ . In the following part, we study its reducibility.

## 4 Reducibility

The reducibility result is based on the fundamental following lemma.

**Lemma 2.** *Let  $\mathcal{U}$  be a proper invariant subspace of  $V_6$ . Then  $\mathcal{U}$  is annihilated by all the  $e_i$ 's. And so, it is also annihilated by all the conjugates of the  $e_i$ 's.*

PROOF. An action by  $e_1$  (resp  $e_2$ , resp  $e_i$ ,  $i \geq 3$ ) on any basis vector always yields a multiple of  $w_{12}$  (resp  $\widehat{w_{23}}$ , resp  $w_{i-1,i}$ ). Moreover, if one of these vectors lies in  $\mathcal{U}$ , then all the other basis vectors are also in  $\mathcal{U}$  and  $\mathcal{U}$  would hence be the whole space. This contradicts  $\mathcal{U}$  is proper.

**Definition 2.** *Define*

$$\begin{aligned} X_{13} &= g_3 e_1 g_3^{-1} & \text{and} & & X_{1j} &= g_j X_{1,j-1} g_j^{-1} & \forall 4 \leq j \leq 6 \\ X_{24} &= g_4 e_3 g_4^{-1} & \text{and} & & X_{2j} &= g_j X_{2,j-1} g_j^{-1} & \forall j \in \{5, 6\} \\ X_{35} &= g_5 e_4 g_5^{-1} & \text{and} & & X_{36} &= g_6 X_{35} g_6^{-1} \\ X_{46} &= g_6 e_5 g_6^{-1} \end{aligned}$$

*Define also*

$$\begin{aligned} \widehat{X_{24}} &= g_4 e_2 g_4^{-1} & \text{and} & & \widehat{X_{2j}} &= g_j \widehat{X_{2,j-1}} g_j^{-1} & \forall 4 \leq j \leq 6 \\ \widehat{X_{3j}} &= g_3 \widehat{X_{2j}} g_3^{-1} & \forall 4 \leq j \leq 6 \\ \widehat{X_{4j}} &= g_4 \widehat{X_{3j}} g_4^{-1} & \forall j \in \{5, 6\} \\ \widehat{X_{56}} &= g_5 \widehat{X_{46}} g_5^{-1} \end{aligned}$$

*Define further*

$$\begin{aligned} \widehat{X_{st}}^{\Delta^+} &= g_1 \widehat{X_{st}} g_1^{-1} & \forall s \geq 3 \\ \widehat{X_{st}}^{\Delta^{++}} &= g_3 \widehat{X_{st}}^{\Delta^+} g_3^{-1} & \forall s \geq 4 \\ \widehat{X_{56}}^{\Delta^{+++}} &= g_4 \widehat{X_{56}}^{\Delta^{++}} g_4^{-1} \\ \widehat{X_{56}}^{\Delta^l} &= g_2 \widehat{X_{56}}^{\Delta^{+++}} g_2^{-1} \end{aligned}$$

*Define finally*

$$X_{12} = e_1 \text{ and } X_{23} = e_3 \text{ and } X_{34} = e_4 \text{ and } X_{45} = e_5 \text{ and } X_{56} = e_6 \text{ and } \widehat{X_{23}} = e_2$$

Let  $S$  denote the sum of all these conjugates.

Suppose  $\nu^{(6)}$  is reducible and let  $\mathcal{U}$  be a proper invariant subspace of  $V_6$ . By Lemma 2, we have

$$S \cdot \mathcal{U} = 0$$

Then, the determinant of the left action by  $S$  must be zero, as  $\mathcal{U}$  is non-trivial. We computed the determinant of this action with Mathematica and we obtained

$$\det(S) = \frac{(l - r^3)^{15} (l + r^3)^{30} (-1 + l r^3)^{20} (1 + l r^9)^6 (-1 + l r^{21})}{l^{36} r^{99} (-1 + r^2)^{36}}$$

This yields a necessary condition for the representation  $\nu^{(6)}$  to be reducible. For the converse, we show the following result.

**Proposition 3.** *The  $F$ -vector space*

$$\begin{aligned} \mathcal{K}_6 = & \bigcap_{1 \leq i < j \leq 6} \text{Ker } \nu^{(6)}(X_{ij}) \cap \bigcap_{2 \leq i < j \leq 6} \text{Ker } \nu^{(6)}(\widehat{X_{ij}}) \cap \bigcap_{3 \leq i < j \leq 6} \text{Ker } \nu^{(6)}(\widehat{X_{ij}}^{\Delta^+}) \\ & \cap \bigcap_{4 \leq i < j \leq 6} \text{Ker } \nu^{(6)}(\widehat{X_{ij}}^{\Delta^{++}}) \cap \text{Ker } \nu^{(6)}(\widehat{X_{56}}^{\Delta^{+++}}) \cap \text{Ker } \nu^{(6)}(\widehat{X_{56}}^{\Delta^l}) \end{aligned}$$

is a  $BMW(E_6)$ -submodule of  $V_6$ .

PROOF. Given any vector  $x$  in  $\mathcal{K}_6$ , by the work already achieved in the proofs of Lemma 5 of [18] and of Lemma 2 in Chapter 2 of [14], it suffices to show that  $g_1 x$  is annihilated by the  $X_{st}$  terms which either carry a hat or a triangle, that  $g_2 x$  is annihilated by all the  $X_{1t}$ 's with  $t \geq 2$  and that when  $i \geq 2$ ,  $g_i x$  is annihilated by all the  $X_{st}$  terms which wear a triangle. First, since  $g_1$  commutes with the  $\widehat{X_{2j}}$ 's, we have  $g_1 x \in \bigcap_{3 \leq j \leq 6} \text{Ker } \nu^{(6)}(\widehat{X_{2j}})$ . To show that  $g_1 x$  is in the kernel of  $\widehat{X_{s,j}}$  with  $s \geq 3$ , it suffices to show that  $g_1^{-1} x$  is in the kernel of  $\widehat{X_{s,j}}$ . Notice

$$\widehat{X_{s,j}} g_1^{-1} x = g_1^{-1} \widehat{X_{s,j}}^{\Delta^+} x = 0$$

So,

$$g_1 x \in \bigcap_{2 \leq i < j \leq 6} \text{Ker } \nu^{(6)}(\widehat{X_{ij}}) \quad (36)$$

Next, by definition of  $\widehat{X_{st}}^{\Delta^+}$ , we have  $g_1 x \in \text{Ker } \nu^{(6)}(\widehat{X_{st}}^{\Delta^+})$  for all  $s \geq 3$ . Fix  $s \geq 4$ .

We now show that  $g_1^{-1}$  commutes with  $\widehat{X_{st}}^{\Delta^{++}}$ . We have

$$\widehat{X_{st}}^{\Delta^{++}} g_1^{-1} x = g_1^{-1} g_1 g_3 \widehat{X_{st}}^{\Delta^+} g_3^{-1} g_1^{-1} x \quad (37)$$

$$= g_1^{-1} g_1 g_3 g_1 \widehat{X_{st}} g_1^{-1} g_3^{-1} g_1^{-1} x \quad (38)$$

$$= g_1^{-1} g_3 g_1 g_3 \widehat{X_{st}} g_3^{-1} g_1^{-1} g_3^{-1} x \quad (39)$$

$$= g_1^{-1} g_3 g_1 \widehat{X_{st}} g_1^{-1} g_3^{-1} x \quad (40)$$

$$= g_1^{-1} \widehat{X_{st}}^{\Delta^{++}} x \quad (41)$$

$$= 0 \quad (42)$$

Equality (40) is easy to see on the tangles after doing a Reidemeister's move of type (III). Further,

$$\widehat{X_{56}}^{\Delta^{+++}} g_1^{-1} x = g_4 g_3 g_1 \widehat{X_{56}} g_1^{-1} g_3^{-1} g_1^{-1} g_4^{-1} x \quad (43)$$

$$= g_1^{-1} g_4 g_3 g_1 g_3 \widehat{X_{56}} g_3^{-1} g_1^{-1} g_3^{-1} g_4^{-1} x \quad (44)$$

$$= g_1^{-1} g_4 g_3 g_1 \widehat{X_{56}} g_1^{-1} g_3^{-1} g_4^{-1} x \quad (45)$$

$$= g_1^{-1} \widehat{X_{56}}^{\Delta^{+++}} x \quad (46)$$

$$= 0 \quad (47)$$

Similarly, we show that

$$\overset{\Delta^l}{X}_{56} g_1^{-1} x = g_1^{-1} \overset{\Delta^l}{X}_{56} x = 0$$

At this stage, we conclude that  $g_1 x \in \mathcal{K}_6$ . Next, we show that  $g_2 x$  is annihilated by all the  $\overset{\Delta}{X}_{1t}$ 's with  $t \geq 2$ . We have  $\overset{\Delta}{X}_{12} g_2 x = 0$  since  $e_1$  commutes with  $g_2$ . Further, when  $t \geq 3$ , we have  $\overset{\Delta}{X}_{1t} = g_1^{-1} \overset{\Delta}{X}_{2t} g_1$ , so that

$$\overset{\Delta}{X}_{1t} g_2 x = g_1^{-1} \overset{\Delta}{X}_{2t} g_2 (g_1 x)$$

We just showed above that  $g_1 x \in \mathcal{K}_6$ . It follows that  $g_2(g_1 x) \in \text{Ker}(\overset{\Delta}{X}_{2t})$  by Lemma 5 of [18].

It remains to show that when  $i \geq 2$ , we have  $g_i x$  belongs to the kernel of the  $\overset{\Delta}{X}_{st}$ 's. We begin with  $i = 2$ . Obviously,  $g_2 x \in \text{Ker } \nu^{(6)}(\overset{\Delta^l}{X}_{56})$  and  $g_2^{-1} x \in \text{Ker } \nu^{(6)}(\overset{\Delta^{+++}}{X}_{56})$ .

Next, for every  $s \geq 3$ , we have

$$\overset{\Delta^+}{X}_{st} g_2^{-1} x = g_1 \widehat{\overset{\Delta^+}{X}_{st}} g_2^{-1} g_1^{-1} x$$

As we have seen above that  $g_1^{-1} x \in \mathcal{K}_6$ , the member to the right hand side of the latter equality is zero. Before we can proceed the other  $\overset{\Delta}{X}$  terms, we must deal with the case  $i = 3$  first. We use the following equalities, with the same techniques as before.

**Lemma 3.**

$$\begin{aligned} \overset{\Delta^+}{X}_{st} g_3^{-1} x &= \begin{cases} g_3^{-1} \overset{\Delta^{++}}{X}_{st} x & \text{if } s \geq 4 \\ g_1 g_3 g_1^{-1} \widehat{\overset{\Delta^+}{X}_{2t}} g_3^{-1} g_1^{-1} x & \text{if } s = 3 \end{cases} \\ \overset{\Delta^{++}}{X}_{st} g_3 x &= g_3 \overset{\Delta^+}{X}_{st} x \\ \overset{\Delta^{+++}}{X}_{56} g_3^{-1} x &= g_3^{-1} \overset{\Delta^{+++}}{X}_{56} x \\ \overset{\Delta^l}{X}_{56} g_3^{-1} x &= g_3^{-1} \overset{\Delta^l}{X}_{56} x \end{aligned}$$

The second part of the first equality is obtained as follows.

$$\begin{aligned} \overset{\Delta^+}{X}_{3t} g_3^{-1} x &= g_1 g_3 \widehat{\overset{\Delta^+}{X}_{2t}} g_3^{-1} g_1^{-1} g_3^{-1} x \\ &= g_1 g_3 \widehat{\overset{\Delta^+}{X}_{2t}} g_1^{-1} g_3^{-1} g_1^{-1} x \\ &= g_1 g_3 g_1^{-1} \widehat{\overset{\Delta^+}{X}_{2t}} g_3^{-1} g_1^{-1} x \end{aligned}$$

Because we have seen earlier that  $g_1^{-1} x \in \mathcal{K}_6$ , we know that  $g_3^{-1}(g_1^{-1} x) \in \text{Ker}(\widehat{\overset{\Delta^+}{X}_{2t}})$ . Thus, all the results above are zero. This ends the case  $i = 3$ . It is

now easy to close the case  $i = 2$ . Indeed, we have for every  $s \geq 4$ ,

$$\widehat{X_{st}}^{\Delta^{++}} g_2^{-1} x = g_3 g_1 \widehat{X_{st}} g_2^{-1} g_1^{-1} g_3^{-1} x$$

By the case  $i = 3$ , we have  $g_3^{-1} x \in \mathcal{K}_6$ . Then by the case  $i = 1$ , we get  $y = g_1^{-1} g_3^{-1} x \in \mathcal{K}_6$ . It follows that  $g_2^{-1} y \in \text{Ker}(\widehat{X_{st}})$ .

We now deal with the case  $i = 4$ . When  $i = 4$ , the following equalities hold.

**Lemma 4.**

$$\begin{aligned} \widehat{X_{st}}^{\Delta^{++}} g_4^{-1} x &= \begin{cases} g_4^{-1} \widehat{X_{56}}^{\Delta^{+++}} x & \text{if } s = 5 \text{ and } t = 6 \\ g_3 g_4 g_1 g_3 g_1^{-1} \widehat{X_{2t}} g_3^{-1} g_4^{-1} g_1^{-1} g_3^{-1} x & \text{if } s = 4 \end{cases} \\ \widehat{X_{56}}^{\Delta^{+++}} g_4 x &= g_4 \widehat{X_{56}}^{\Delta^{+++}} x \\ \widehat{X_{56}}^{\Delta^+} g_4^{-1} x &= g_2 g_4 g_3 g_1 \widehat{X_{56}} g_2^{-1} g_1^{-1} g_3^{-1} g_4^{-1} g_2^{-1} x \end{aligned} \quad (48)$$

$$\widehat{X_{st}}^{\Delta^+} g_4^{-1} x = g_4^{-1} g_1 g_4 \widehat{X_{st}} g_4^{-1} g_1^{-1} x = \begin{cases} g_4^{-1} \widehat{X_{34}}^{\Delta^+} x & \text{if } s = 3 \text{ and } t = 4 \\ g_4^{-1} \widehat{X_{4t}}^{\Delta^+} x & \text{if } s = 3 \text{ and } t \geq 5 \\ g_4^{-1} \widehat{X_{56}}^{\Delta^+} x & \text{if } s = 5 \text{ and } t = 6 \end{cases}$$

$$\widehat{X_{4t}}^{\Delta^+} g_4 x = g_4 \widehat{X_{3t}}^{\Delta^+} x \quad \forall t \in \{5, 6\}$$

To get the second term in the first equality, proceed as follows.

$$\begin{aligned} \widehat{X_{4t}}^{\Delta^{++}} g_4^{-1} x &= g_3 g_1 g_4 \widehat{X_{3t}} g_4^{-1} g_1^{-1} g_3^{-1} g_4^{-1} x \\ &= g_3 g_4 g_1 g_3 \widehat{X_{2t}} g_3^{-1} g_1^{-1} g_3^{-1} g_4^{-1} g_3^{-1} x \\ &= g_3 g_4 g_1 g_3 \widehat{X_{2t}} g_1^{-1} g_3^{-1} g_1^{-1} g_4^{-1} g_3^{-1} x \\ &= g_3 g_4 g_1 g_3 g_1^{-1} \widehat{X_{2t}} g_3^{-1} g_4^{-1} g_1^{-1} g_3^{-1} x \end{aligned}$$

It is now easy to conclude. By Lemma 3, we know that  $g_3^{-1} x \in \mathcal{K}_6$ . Then,  $y = g_1^{-1}(g_3^{-1} x)$  is also in  $\mathcal{K}_6$ . In particular,  $y$  belongs to the kernel of  $X_{ij}$  and of  $\widehat{X_{ij}}$  for all the integers  $i$  and  $j$  with  $2 \leq i < j \leq 6$ . Then, an adequate adaptation of Lemma 5 of [18] shows that

$$g_3^{-1} g_4^{-1} y \in \text{Ker}(\widehat{X_{2t}})$$

To show that the right member of equality (48) is zero, it suffices to show that

$$g_1^{-1} g_3^{-1} g_4^{-1} g_2^{-1} x \in \cap_{2 \leq i < j \leq 6} \text{Ker}(X_{ij} \cap \widehat{X_{ij}})$$

For convenience, denote by  $\overline{\mathcal{K}_5}$  this intersection. First, we have  $g_2^{-1} x \in \mathcal{K}_6$ . By the equalities of Lemma 4, it follows that  $g_4^{-1} g_2^{-1} x$  belongs to the kernel

of all the  $X$ 's, except possibly  $\overset{\Delta^i}{X}_{56}$ . Then,  $g_1^{-1}(g_4^{-1}g_2^{-1}x) \in \overline{\mathcal{K}_5}$ . It follows that  $g_3^{-1}g_4^{-1}g_2^{-1}x$  is in the kernel of all the  $X$ 's, except possibly  $\overset{\Delta^i}{X}_{56}$ . Then,  $g_1^{-1}g_3^{-1}g_4^{-1}g_2^{-1}x$  belongs to  $\overline{\mathcal{K}_5}$  as desired. This ends the case  $i = 4$ .

**Lemma 5.**

$$\begin{aligned}
\overset{\Delta^+}{X}_{35} g_5 x &= g_5 \overset{\Delta^+}{X}_{34} x \\
\overset{\Delta^+}{X}_{56} g_5 x &= g_5 \overset{\Delta^+}{X}_{46} x \\
\overset{\Delta^+}{X}_{34} g_5^{-1} x &= g_5^{-1} \overset{\Delta^+}{X}_{35} x \\
\overset{\Delta^+}{X}_{46} g_5^{-1} x &= g_5^{-1} \overset{\Delta^+}{X}_{56} x \\
\overset{\Delta^+}{X}_{st} g_5^{-1} x &= g_5^{-1} \overset{\Delta^+}{X}_{st} x \quad \text{if } (s, t) \in \{(4, 5), (3, 6)\} \\
\overset{\Delta^{++}}{X}_{56} g_5 x &= g_5 \overset{\Delta^{++}}{X}_{46} x \\
\overset{\Delta^{++}}{X}_{45} g_5 x &= g_5 \overset{\Delta^{++}}{X}_{45} x \\
\overset{\Delta^{++}}{X}_{46} g_5^{-1} x &= g_5^{-1} \overset{\Delta^{++}}{X}_{56} x \\
\overset{\Delta^{+++}}{X}_{56} g_5^{-1} x &= g_4 g_5 g_3 g_4 g_1 g_3 g_1^{-1} \widehat{X_{26}} g_3^{-1} g_4^{-1} g_5^{-1} g_1^{-1} g_3^{-1} g_4^{-1} x \quad (49) \\
\overset{\Delta^i}{X}_{56} g_5^{-1} x &= g_2 \overset{\Delta^{+++}}{X}_{56} g_5^{-1} g_2^{-1} x \quad (50)
\end{aligned}$$

Equality (49) is obtained by using the expression for  $\overset{\Delta^{++}}{X}_{46} g_4^{-1}$  from Lemma 4.

With (50), the fact that  $\overset{\Delta^i}{X}_{56} g_5^{-1} x$  is zero follows from (49) and from the fact that  $g_2^{-1} x \in \mathcal{K}_6$ . It remains to deal with  $i = 6$ .

**Lemma 6.**

$$\begin{aligned}
\overset{\Delta^l}{X_{56}} g_6 x &= g_6 \overset{\Delta^l}{X_{56}} x \\
\overset{\Delta^{+++}}{X_{56}} g_6 x &= g_6 \overset{\Delta^{+++}}{X_{56}} x \\
\overset{\Delta^{++}}{X_{45}} g_6^{-1} x &= g_6^{-1} \overset{\Delta^{++}}{X_{46}} x \\
\overset{\Delta^{++}}{X_{s,t}} g_6 x &= \begin{cases} g_6 \overset{\Delta^{++}}{X_{56}} x & \text{if } s = 5 \text{ and } t = 6 \\ g_6 \overset{\Delta^{++}}{X_{45}} x & \text{if } s = 4 \text{ and } t = 6 \end{cases} \\
\overset{\Delta^+}{X_{s,5}} g_6^{-1} x &= g_6^{-1} \overset{\Delta^+}{X_{s,6}} x \quad \forall s \in \{3, 4\} \\
\overset{\Delta^+}{X_{s,6}} g_6 x &= g_6 \overset{\Delta^+}{X_{s,5}} x \quad \forall s \in \{3, 4\} \\
\overset{\Delta^+}{X_{st}} g_6 x &= g_6 \overset{\Delta^+}{X_{st}} x \quad \text{if } (s, t) \in \{(3, 4), (5, 6)\}
\end{aligned}$$

This achieves the proof of Proposition 3. A consequence of Lemma 2 and Proposition 3 is that when the representation is reducible, any proper invariant subspace of  $V_6$  is a  $BMW(E_6)$ -submodule of  $\mathcal{K}_6$ . The following proposition studies the dimension of  $\mathcal{K}_6$  as a vector space over  $F$ .

**Proposition 4.** *Ker  $S \subseteq \mathcal{K}_6$ . Thus,  $\text{Ker } S = \mathcal{K}_6$  and  $\dim(\mathcal{K}_6) = 36 - \text{rank}(S)$ .*

PROOF. Straightforward computations show that an action by  $e_1$  on any vector is always a multiple of  $w_{12}$  and an action by  $e_2$  on any vector is always a multiple of  $\widehat{w_{23}}$ . It follows that an action by  $X_{st}$  with  $1 \leq s < t \leq 6$  on any vector is always proportional to  $w_{st}$  and an action by  $\widehat{X_{st}}$  with  $2 \leq s < t \leq 6$  on any vector is always proportional to  $\widehat{w_{st}}$ . Further, by definition of the  $X$ 's

which carry a triangle, an action by  $\overset{\Delta^+}{X_{st}}$  (resp  $\overset{\Delta^{++}}{X_{st}}$ ) with  $s \geq 3$  (resp  $s \geq 4$ ) on any vector is always proportional to  $\overset{\Delta^+}{w_{st}}$  (resp  $\overset{\Delta^{++}}{w_{st}}$ ); an action by  $\overset{\Delta^{+++}}{X_{56}}$  (resp  $\overset{\Delta^l}{X_{56}}$ ) on any vector always yields a multiple of  $\overset{\Delta^{+++}}{w_{56}}$  (resp  $\overset{\Delta^l}{w_{56}}$ ).

In other words, each line of the matrix  $S$  corresponds to the action of exactly one conjugate from Definition 2.

We computed with Maple the rank of the sum matrix  $S$  for the values of  $l$  and  $r$  that annihilate the determinant. We found the following values.

$$\begin{aligned}
\text{When } l = r^3 \quad rk(S) &= 21 \\
\text{When } l = -r^3 \quad rk(S) &= 6 \\
\text{When } l = \frac{1}{r^3} \quad rk(S) &= 16 \\
\text{When } l = -\frac{1}{r^9} \quad rk(S) &= 30 \\
\text{When } l = \frac{1}{r^{21}} \quad rk(S) &= 35
\end{aligned}$$

The respective dimensions above correspond to the exponents in the determinant. In particular, for each of these values of  $l$  and  $r$ , the module  $\mathcal{K}_6$  is non-trivial. Since it is not the whole space  $V_6$  either, this shows that for these values the representation  $\nu^{(6)}$  is reducible. The necessary and sufficient conditions are gathered in the following proposition.

**Proposition 5.** *The representation  $\nu^{(6)}$  is reducible if and only if*

$$l \in \left\{ r^3, -r^3, \frac{1}{r^3}, -\frac{1}{r^9}, \frac{1}{r^{21}} \right\}$$

We conclude the proof of Theorem 1 by noticing that

$$\nu^{(6)}(e_i e_j) = 0 \quad \text{when } i \not\sim j$$

This is simply because our representation  $\nu^{(6)}$  is built inside the  $B(E_6)$ -module  $M \otimes_{\mathcal{H}} F$  that was introduced earlier. Then,  $\nu^{(6)}$  is an irreducible representation of

$$B(E_6)e_1 B(E_6) / < B(E_6)e_i e_j B(E_6) >_{i \not\sim j}$$

This quotient of ideals is called  $I_1/I_2$  in [5]. As part of their work in [5], for each irreducible representation of the Hecke algebra of type  $A_5$  of degree  $d$ , Arjeh Cohen, Dié Gijsbers and David Wales build an irreducible representation of  $I_1/I_2$  of degree  $36d$ . They show all the inequivalent irreducible representations of  $I_1/I_2$  are obtained this way. There are only two inequivalent representations of the Hecke algebra of type  $A_5$  of degree 1. Thus, there are only two inequivalent irreducible representations of  $I_1/I_2$  of degree 36. The representation that we built in this paper is one of them. Our  $r$  is the  $\frac{1}{r}$  of [5]. Moreover, up to the change of parameters

$$t = \frac{1}{lr^3}$$

and up to some rescaling of the generators, the representation of [5] is equivalent, as a representation of the Artin group of type  $E_6$ , to the representation of [4] that was defined by the authors and used by them to show the linearity of the Artin group. We called the latter representation the Cohen-Wales representation of type  $E_6$ . We get Theorem 1.

A consequence of Lemma 2 is that when the representation is reducible, it is indecomposable. Then the BMW algebra is not semisimple. So, for the values of  $l$  and  $r$  of Proposition 5, the algebra is not semisimple. Since  $r$  and  $-\frac{1}{r}$  play identical roles, we obtain a second set of values of the parameters for which the algebra is not semisimple, as in the statement of Theorem 2.

We conclude this paper by noting that the method that we use allows to find Hecke algebra representations of type  $E_6$ .

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